2
Basic Modeling Concepts in Environmental Systems Models

Chapter Objectives—
After you finish this chapter, you should be able to:

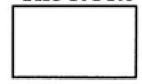
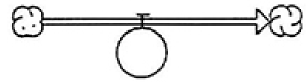
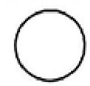
1. *Describe five* ***behavior patterns*** *that are present in many dynamic systems. These five patterns serve as the building blocks for many complicated systems models. The five* ***behavior patterns*** *are:*
   - *Linear growth or decay*
   - *Exponential growth or decay*
   - *Logistic growth*
   - *Overshoot and collapse*
   - *Oscillation*
2. *Construct and recognize systems diagrams for systems exhibiting each type of behavior.*
3. *Describe the underlying mathematical relationships behind each type.*
4. *Identify the types of feedback required to create each type of behavior and determine conditions under which each pattern achieves steady-state behavior.*
5. *Identify environmental systems where each type of behavior exists.*

2.1 Introduction: Building Blocks for Environmental Systems Models

In Chapter 1 we discussed an imaginary construction project in which you were going to build a new home. Let us suppose that your contractor has the building plans and building materials (i.e., lumber, nails, cement, etc.) and is now ready to begin construction. The builder will combine these materials to construct your home.

Building an environmental systems model is much like building a house. You need some "building materials." The building materials for a systems model, fortunately, are much simpler than those used for building a house. There are only four basic building blocks. These are the four system components introduced in Section 1.2. Those same components are reproduced in Table 2.1 for your reference. In addition, we have displayed the symbol used to represent each component. These symbols are not universal, but they do reflect modeling conventions used in this text. Every environmental system described in this text can be modeled using only these four "build-

TABLE 2.1. Four systems components and their modeling symbols.

Name	Description	Symbol
Reservoir	A component of a system where something is accumulated. The contents of the reservoir may go up or down over time.	The ***stock*** 
Processes	Activities that determine the values of reservoirs over time.	The ***flow*** 
Converters	System quantities that dictate the rates at which the processes operate and the reservoirs change.	The ***converter*** 
Interrelationships	Define the cause-effect relationships between system elements.	The ***connector*** 

ing blocks." In fact, models of very complex systems can be built from these four simple components.

If a homebuilder has access to the right building materials, there is still no guarantee that he or she will build a home that will meet your needs or stand the test of time. The contractor also needs to use good construction principles. For example, the foundation must be sufficiently deep and the floor joists need to be large enough to support the weight of the floor and the occupants.

In the same way, using the four building blocks in Table 2.1 does not guarantee that any model built with them will be correct or even useful. Certain rules of construction must also be followed. We will now outline five rules for building a systems model that must be followed if your model is to be useful and reliable. These have been illustrated in Chapter 1. We state them explicitly here for your reference, and will follow these rules throughout the text to guide our modeling activities. Other modeling principles will be given later to assure that your model is ***valid*** (i.e., that it accurately represents the real-life system) and is useful for addressing the questions you want to answer.

Rules for Building Systems Models

1. Keep the systems diagram (and hence the model) as simple as possible. Add complexity only as it is needed.
2. Use common-sense mathematical expressions to define the relationships between elements in the system.

3. If a credible mathematical expression is not readily available, define relationships by using graphs.
4. Make sure you identify the units in which time is to be measured in the model. Also identify the units of measurement used for each element in the system, and make sure that the units are compatible with the mathematical expressions you defined under Point 2.
5. Be sure that the only system entities that directly affect the values of a particular reservoir are the inflows and outflows associated with that reservoir.

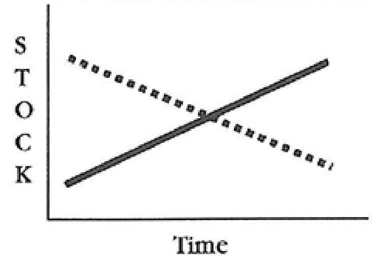
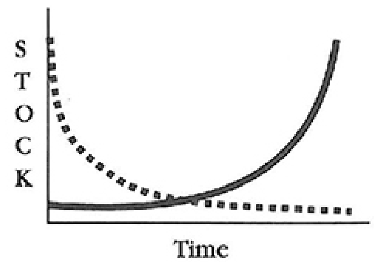
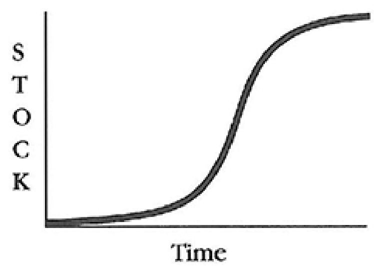
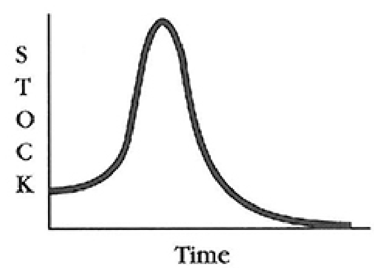
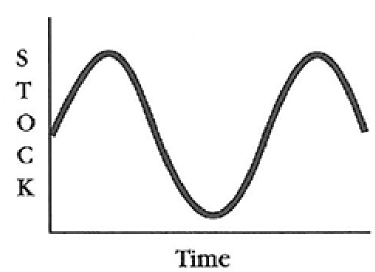
Let us now use our home building example to illustrate how the rest of this chapter is organized. Builders seldom construct a home completely from "scratch." For example, even though the doors in the home are made from wood, the builder does not actually construct the door on the building site. A builder usually purchases a prefabricated door and frame. In fact, builders have even developed names for the various sizes and styles of door units. By examining the architect's drawings, the builder identifies how many door units of each type are needed and then orders them from the factory. This is much more efficient (and much less expensive!) than building the doors from scratch.

The same is true about systems models. Many environmental systems have common features. These features tend to lead to certain types of behaviors that can be predicted and easily modeled. Much of the art behind environmental modeling consists of recognizing these features and then using the proper modeling constructs (i.e., the right combinations of stocks, flows, converters, and connectors) to model them. We will refer to these commonly occurring features as ***behavior patterns***. By calling something a behavior pattern, we are simply referring to a common pattern of behavior exhibited by one or more of the system reservoirs. The main body of this chapter is devoted to defining and illustrating the five common behavior patterns described in Table 2.2. For each type, the discussion will follow this outline:

1. Illustrative example. We will employ an example of a simple system having one or more reservoirs that exhibit that particular type of behavior. System diagrams and underlying mathematical relationships for the examples will be provided.

2. System features, diagram, **and equations.** We will identify the underlying features that are necessary for a system to exhibit this type of behavior. A generic system diagram and the underlying mathematical relationships will be given. We will identify the types of feedback present in each case. We will also borrow from the tools of calculus to develop equations that describe the *rate* at which a reservoir changes whenever it exhibits the type of behavior discussed. These equations are called ***rate equations***. They will play an important role in describing dynamic systems and in understanding and predicting how such systems behave. The rate equations will

TABLE 2.2. Five common behavior patterns in dynamic systems.

Type of dynamic	Behavior over time
1. Linear growth or decay	
2. Exponential growth or decay	
3. Logistic growth	
4. Overshoot and collapse	
5. Oscillation	

also be used to determine if the system will reach a steady state and under what conditions it will do so.

3. Summary table. We will provide a table to highlight the distinguishing characteristics of each type of behavior and to identify the conditions under which each type of behavior can occur.

By covering all five types of behaviors, you will develop a set of prefabricated "modeling constructs" that you can use (like prefabricated doors) to help you understand and build environmental systems models.

2.2 Behavior Pattern #1: Linear Growth or Decay

2.2.1 Linear Growth or Decay: Illustrative Example

Consider an underground reserve of 10 million barrels of oil that is being consumed at a fixed rate of 10,000 barrels per day. A diagram for this simple system is shown in Figure 2.1. Note that time is measured in days in this model.

It should be obvious that the *Oil Reserves* will begin with a value of 10 million barrels at time zero and will decrease by 10,000 barrels each day until it is finally depleted after 1,000 days. In fact, the equation for calculating the size of the *Oil Reserves* one day ahead is (see Equation 1.1):

Oil Reserves Tomorrow = *Oil Reserves Today* − 10,000 barrels

Note that the preceding equation is applicable only as long as the *Oil Reserves* reservoir has a positive value. In other words, the equation is only applicable for 1,000 days, after which the *Oil Reserves* will be a constant value of zero because it cannot be negative.

If we wanted to know the size of the *Oil Reserves* 12 hours from now, we would subtract only 5,000 barrels (because 12 hours is one half of a day). If we wanted to know the size of the *Oil Reserves* at a point in time n days from now, we would subtract n*10,000 barrels. We can write this in a more general form by letting $R(t)$ represent the size of the *Oil Reserves* at time t (where t is measured in days). The ***difference equation*** for calculating the size of the *Oil Reserves* at a future time $t + \Delta t$ is given by Equation (2.1). A plot of the behavior of the *Oil Reserves* over time is given in Figure 2.2.

$$R(t + \Delta t) = R(t) - (10{,}000 \cdot \Delta t) \qquad (2.1)$$

In the *Oil Reserves* example, there is a single outflow process that has a constant value of −10,000 barrels/day until the reservoir is depleted. The rate of decrease is equal to the value of the outflow process. This rate is depicted graphically as the slope of the line in Figure 2.2.

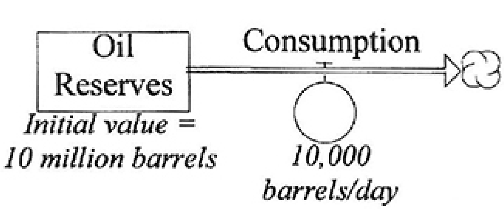


FIGURE 2.1. Oil consumption model.

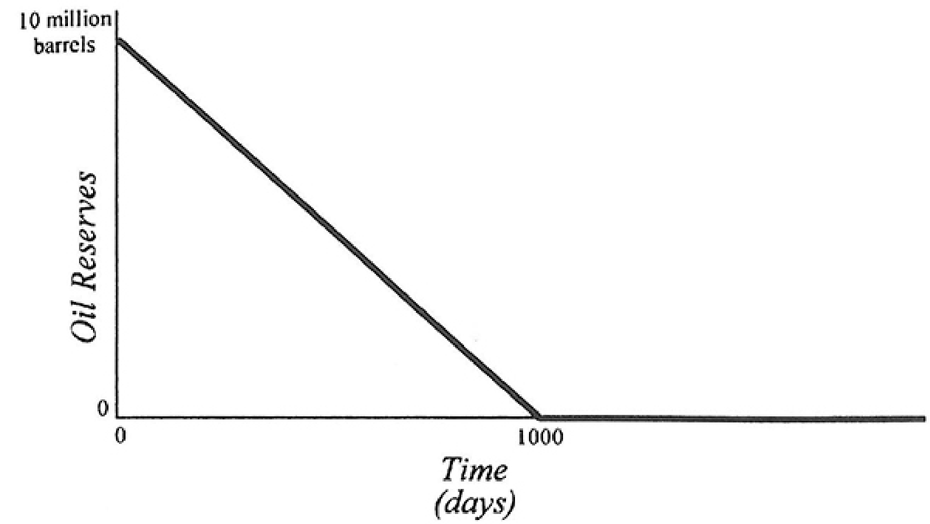


FIGURE 2.2. Oil reserves versus time.

The system in Figures 2.1 and 2.2 (prior to t = 1,000 days) illustrates the ***linear decay*** behavior pattern. A system undergoing ***linear growth*** would also follow a straight-line plot, but would slope upward instead of downward. In order for a reservoir to exhibit linear growth or decay, *the sum of all the inflows into the reservoir, minus the sum of all its outflows must be constant.* This can happen whenever each of the inflow and outflow processes affecting the reservoir is constant. In addition, one of the end-of-chapter problems demonstrates that linear growth or decay can also occur even if some of the inflows and outflows are not constant.

2.2.2 Linear Growth or Decay: System Features, Diagram, and Equations

A linear behavior pattern is one in which the reservoir of interest changes at a constant rate over time. In order for a reservoir to exhibit linear growth or decay, therefore, the sum of all the inflows into the reservoir, minus the sum of all its outflows must be constant. If the constant is positive, then the system will display linear growth. If the constant is negative, then the system will exhibit linear decay. If the constant is equal to zero, then the reservoir will remain constant through time. A generic linear system with multiple inflows and outflows is shown in Figure 2.3.

Note that such a system can have any number of inflows and outflows. In addition, it is not necessary for each of the flows to be constant through time in order for the system to exhibit linear growth or decay. It is necessary, however, that the sum of the inflows, minus the sum of the outflows be constant. In other words,

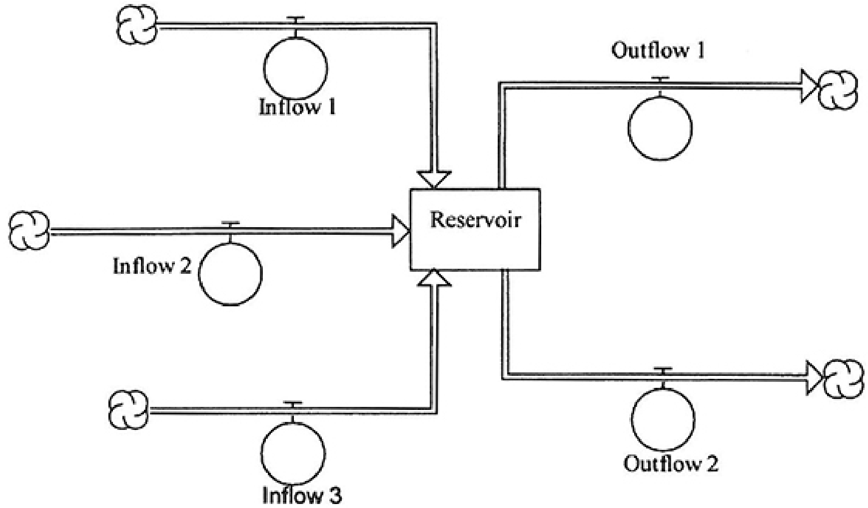


FIGURE 2.3. Generic system diagram for linear growth or decay.

Linear growth or decay occurs if and only if changes in the reservoir over an interval from time t to $t + \Delta t$ are constant for all t. If this change is positive, the reservoir will exhibit ***linear growth***. If the change is negative, the reservoir will exhibit ***linear decay***.

This property is demonstrated for the *Oil Reserves* model. A simple rearrangement of the terms in Equation (2.1) yields the following expression for the total net change in the *Oil Reserves* in a time interval of Δt days.

$$R(t + \Delta t) - R(t) = -10{,}000 \cdot \Delta t \tag{2.2}$$

Because this change does not depend on it and is negative the *Oil Reserves* model exhibits linear decay for 1,000 days. After $t = 1{,}000$ days, $R(t)$ remains at a value of zero.

A careful examination of the system diagram in Figure 2.3 demonstrates that the simple linear system does not contain any kind of feedback. Remember that a feedback loop will always cause an initial change in the system to eventually be either damped out (in the case of ***counteracting feedback***) or amplified (in the case of ***reinforcing feedback***). Because a linear system changes at a constant rate, no damping or amplification can occur; therefore, feedback is absent.

To generalize the linear model mathematically, let $R(t)$ stand for the value of the reservoir at time t in Figure 2.3. The difference equation for $R(t)$ for this generic system is given by the following:

$$R(t + \Delta t) = R(t) + \{(\text{Inflow}_1 + \text{Inflow}_2 + \text{Inflow}_3) - (\text{Outflow}_1 + \text{Outflow}_2)\} \cdot \Delta t \tag{2.3}$$

Because all of the inflows and outflows in this equation are constant values, the change in the reservoir from time t to time $t + \Delta t$ (given by the bracketed term on the right side of Equation (2.3)) will also be constant. This equation can be generalized for any number of inflow or outflow processes.

We understand from calculus that the derivative of $R(t)$ with respect to time stands for the rate at which the value of $R(t)$ changes over time. Hence, the derivative of any reservoir exhibiting a linear behavior pattern must be a constant; therefore, if we let $\frac{dR}{dt}$ stand for the derivative of the reservoir with respect to time, the linear decay model can be represented as:

$$\frac{dR(t)}{dt} = k \tag{2.4}$$

where $R(t)$ is the value of the reservoir at time t, and where k is a constant. If k is positive, the system will exhibit linear growth. If k is negative, the system will exhibit linear decay. In addition, the value of k is also equal to the slope of the graph of $R(t)$ versus time, as depicted in Figure 2.4.

The rate constant k also has a direct relationship to the inflow and outflow processes in Figure 2.3. From the difference Equation (2.3) we can write

$$R(t+\Delta t) - R(t) = \{(\text{Inflow}_1 + \text{Inflow}_2 + \text{Inflow}_3) - (\text{Outflow}_1 + \text{Outflow}_2)\} \cdot \Delta t$$

Dividing both sides by Δt gives the following.

$$\frac{R(t+\Delta t) - R(t)}{\Delta t} = \{(\text{Inflow}_1 + \text{Inflow}_2 + \text{Inflow}_3) - (\text{Outflow}_1 + \text{Outflow}_2)\}$$

By taking the limit of both sides as Δt goes to zero, we obtain the derivative of $R(t)$ as follows.

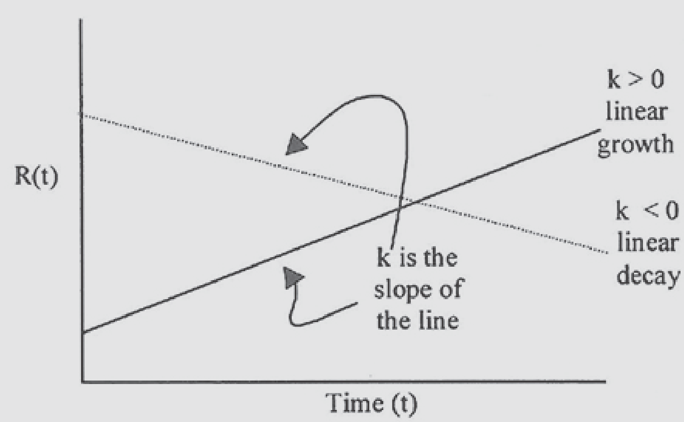


FIGURE 2.4. Linear growth or decay: $\frac{dR(t)}{dt} = k$.

$$\lim_{\Delta t \to 0} \frac{R(t+\Delta t)-R(t)}{\Delta t} = \frac{dR(t)}{dt} = \{(\text{Inflow}_1 + \text{Inflow}_2 + \text{Inflow}_3) - (\text{Outflow}_1 + \text{Outflow}_2)\} \quad (2.5)$$

A comparison of Equation (2.4) with Equation (2.5) reveals that the rate constant k is equal to $\{(\text{Inflow}_1 + \text{Inflow}_2 + \text{Inflow}_3) - (\text{Outflow}_1 + \text{Outflow}_2)\}$. In general, when there are several inflow and outflow processes for a system exhibiting linear growth or decay, the following relationship holds between the rate constant k in Equation (2.4) and the inflow and outflow processes:

$$k = (\text{sum of all inflows}) - (\text{sum of all outflows}) \quad (2.6)$$

Consider the ***Oil Reserves*** example in Figure 2.1. Because there is only one outflow and no inflows into the reservoir, the right side of Equation (2.6) simplifies to

$$k = -10{,}000 \text{ barrels/day.}$$

This implies that the ***Oil Reserves*** reservoir will follow a linear decay. The graph of the ***Oil Reserves*** versus time will display a straight line with a downward slope equal to −10,000 barrels/day.

In order to analyze the steady-state behavior of a linear system, we will give a more detailed analysis of the derivative of the reservoir in such a system. Recall from Section 1.4.5 that whenever a reservoir $R(t)$ exhibits steady-state behavior, its derivative with respect to time will be zero; that is, $\frac{dR(t)}{dt} = 0$. For the linear growth/decay system, we have already shown in Equations (2.4) and (2.6) that $\frac{dR(t)}{dt}$ is a constant value k, where k is equal to the difference in all inflows minus all outflows associated with that reservoir. A linear system, therefore, cannot achieve steady state unless $k = 0$. This occurs only if the sum of all inflows minus the sum of all outflows is zero. This happens only if the system is ***always*** at steady state. The only exception to this is whenever there is some external constraint on the reservoir that does not allow it to either exceed or fall below some predetermined value. In the case of the ***Oil Reserves*** example, we know ahead of time that the ***Oil Reserves*** cannot fall below zero. Because all of the original ***Oil Reserves*** are depleted after day 1,000, the ***Oil Reserves*** must necessarily remain at a constant value of zero after that time. In this case, the ***Oil Reserves*** are not at a steady state for the first 1,000 days and are then at a steady state after 1,000 days.

2.2.3 Linear Growth and Decay: Summary

Table 2.3 shows a summary of the linear growth and decay behavior pattern. You may have noticed that this table includes something called the ***solution***

TABLE 2.3. Defining characteristics of linear growth or decay.

Description	The reservoir increases at a constant rate (linear growth) or decreases at a constant rate (linear decay).
Rate equation	$\frac{dR(t)}{dt} = k$, where k is a constant. • k is the slope of the graph of $R(t)$. • $k > 0$ leads to linear growth; $k < 0$ leads to linear decay • k gives the ***net change*** in the value of $R(t)$ per unit of time. • k = (sum of all the inflows) – (sum of the outflows)
Solution to the rate equation	$R(t) = R_0 + kt$, where R_0 is the initial value of $R(t)$ at time $t = 0$.
Graphical behavior	Graph of $R(t)$ versus t is a straight line
Steady-state behavior	None, unless $k = 0$, or (in the case of linear decay) the reservoir reaches zero and cannot go below zero.
Example applications	Resource management with constant inflows and outflows

to the rate equation. This represents the mathematical expression for $R(t)$ that satisfies the requirements implied by the rate equation.

The rate equation for a linear growth/decay system says that the rate at which the reservoir $R(t)$ changes is constant. This means that a graph of $R(t)$ versus time must follow a straight line. Hence, we know that $R(t)$ must have the form

$$R(t) = a + bt \tag{2.7}$$

where b is the rate of change and a is the value of $R(t)$ at time $t = 0$. Substituting k for b and R_0 for a in Equation (2.7) gives the expression in Table 2.3.

2.3 Behavior Pattern #2: Exponential Growth or Decay

2.3.1 Exponential Growth or Decay: Two Illustrative Examples

The linear growth/decay behavior pattern is very simple, but its applicability is limited. Most naturally occurring systems do not change at a constant rate. Two examples are used to illustrate this.

Suppose that a young child in your household brought home a pair of white mice (one male, one female) that her science teacher gave to her. Further imagine that (due to poor game management practices), the mice escaped from their cage. Your family would probably see few (if any) white mice at first. After a period of time, however, these two mice might mate and bear some offspring. Within a short period of time, this second generation

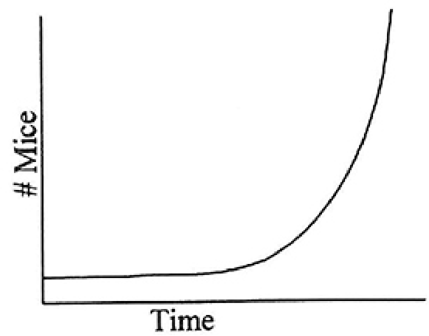


FIGURE 2.5. Population of mice versus time.

of mice would be ready to mate and bear offspring. The number of mice born over time would increase as the number of mice available to bear offspring increased. Hence, the white mouse population would grow slowly at first, and then more rapidly as the number of mice increased. A graph of the number of mice versus time might look something like Figure 2.5.

The graph demonstrates that the rate of growth of the mouse population is not constant. This system does not exhibit linear growth. We will show that this is an example of ***exponential growth***.

The system diagram and equations for the mouse population are given in Figure 2.6. Note that we have specified a birth rate of 1.1 mice per capita per month. This means that (on the average) there will be 11 mice born every month for every 10 mice in the population. This is admittedly a very prolific mouse population! In addition, we have specified a death rate of 0.08 mice per capita per month. This means that 8 of every 100 mice in the population will die each month. It would be important to obtain realistic values of the birth and death rates before using the model to make predictions. We will use these rather arbitrary values right now, however, to illustrate how this model works.

Suppose we let $W(t)$ represent the number of white mice in the population. Based on what we learned in Chapter 1, we know that the equation for

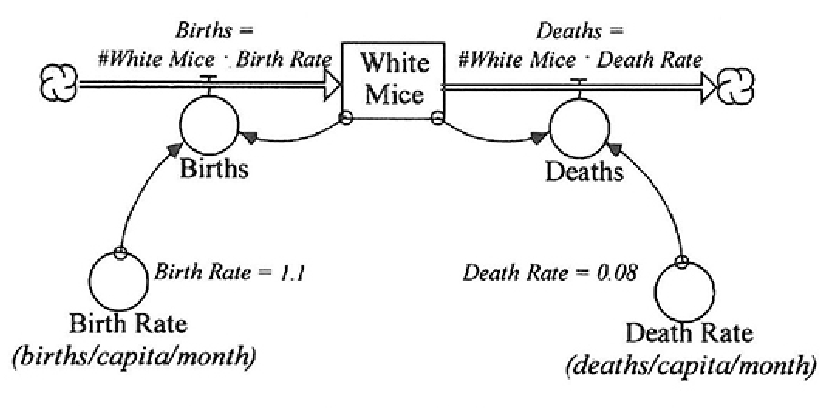


FIGURE 2.6. System diagram for the mouse population.

$W(t + \Delta t)$, the number of white mice at a future time $t + \Delta t$ is determined by the following ***difference equation***.

$$\begin{aligned} W(t+\Delta t) &= W(t) + \{(\text{sum of all inflows}) - (\text{sum of all outflows})\}\cdot \Delta t \\ &= W(t) + \{\textit{Births} - \textit{Deaths}\}\cdot \Delta t \\ &= W(t) + \{\textit{Birth Rate}\cdot W(t) - \textit{Death Rate}\cdot W(t)\}\cdot \Delta t \\ &= W(t) + \{1.1 - 0.08\}\cdot W(t)\cdot \Delta t \\ &= W(t) + \{1.02\}\cdot W(t)\cdot \Delta t \qquad (2.8) \end{aligned}$$

For another example, imagine a perfectly cylindrical bucket full of water, with a small outlet valve at the very bottom (see Figure 2.7). If the valve is opened, water will flow from the bucket onto the ground. The rate at which the water flows out will be greater at the beginning because the pressure of the water still in the bucket pushes down, forcing water out. As the water volume in the bucket decreases, however, the downward pressure lessens. Hence, the rate at which the water will flow from the bucket will also decrease. A graph of the water volume (in cubic centimeters) versus time would look like Figure 2.7.

This is an example of ***exponential decay***. The rate at which the water volume changes is proportional to the amount of water still in the bucket. For example, the water will flow out twice as fast when there are 1,000 cc of water in the bucket as it will when there are 500 cc of water. Over time, the remaining water volume decreases more slowly and eventually (***asymptotically***) approaches zero. The system diagram for this example is given in Figure 2.8. Note that we have again picked an arbitrary value for the *Flow Rate*. This value can (and should!) be determined from other research or even from experimentation, if necessary. We use this arbitrary value now only to illustrate the behavior of the model. The Flow Rate is expressed as the volume of water exiting the bucket per second (in cubic centimeters per second) for every cubic centimeter of water still in the bucket. The specific value of 0.1 cc/sec/cc indicates that if there are 100 cc of water in the bucket, then 10 cc will flow out over the next second. If there are only 10 cc of water in the bucket, then only 1 cc will flow out over the next second. This

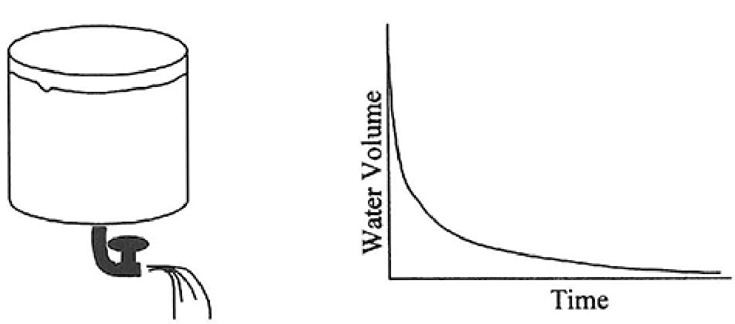


FIGURE 2.7. Exponential decay water example.

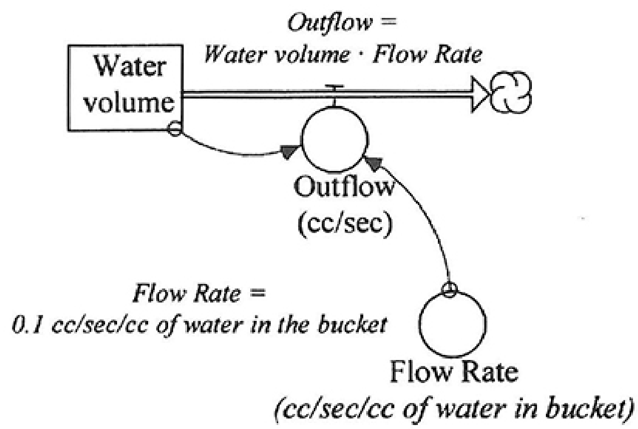


FIGURE 2.8. System diagram for the water bucket.

corresponds to the original description of this system in which it was noted that water flows out at a slower rate whenever there is less water in the bucket to help "push" it out.

These two examples illustrate the fundamental feature of systems exhibiting exponential growth or decay.

Exponential growth (or decay) occurs if and only if the reservoir increases (or decreases) at a rate that is proportional to its size. If the reservoir is increasing in size, then the system exhibits ***exponential growth***. If the reservoir is decreasing in size, then the system exhibits ***exponential decay***.

2.3.2 Exponential Growth or Decay: System Features, Diagram, and Equations

Figure 2.9 shows a generic example of a simple exponential growth/decay system. Make sure that you can identify the differences between this type of system and the simple linear system displayed in Figure 2.3. In particular, the flow processes in Figure 2.9 all operate at rates that are proportional to the current size of the reservoir. This is different than the linear system, where

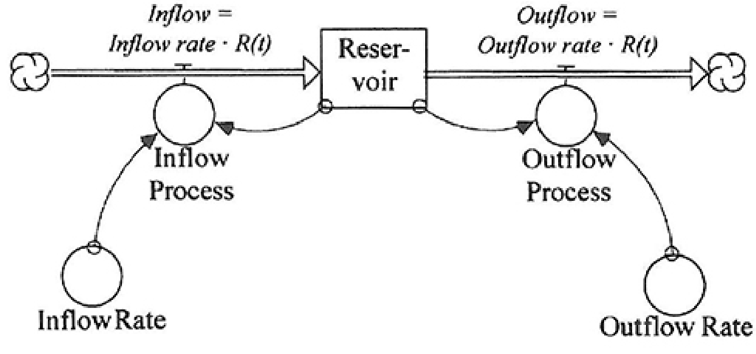


FIGURE 2.9. Generic system diagram for exponential growth or decay.

the net rate of the inflow and outflow processes is constant, independent of the size of the reservoir.

In addition, note that the exponential system has ***feedback loops***. Figure 2.9 has two such loops. The first loop includes the inflow process and the reservoir. If the inflow process increases, then the reservoir will also increase, which in turn drives the inflow to a higher level. This is a ***reinforcing (positive) feedback*** loop. The other loop in Figure 2.9 involves the reservoir and the outflow process. This loop is a ***counteracting (negative) feedback*** loop.

The ***difference equation*** for the reservoir $R(t)$ in Figure 2.9 is

$$\begin{aligned} R(t+\Delta t) &= R(t) + \{\textit{Inflow Rate} \cdot R(t) - \textit{Outflow Rate} \cdot R(t)\} \cdot \Delta t \\ &= R(t) + \{\textit{Inflow Rate} - \textit{Outflow Rate}\} \cdot R(t) \cdot \Delta t \end{aligned} \tag{2.9}$$

The term $\{\textit{Inflow Rate} - \textit{Outflow Rate}\} \cdot R(t)\Delta t$ on the right side of Equation (2.9) demonstrates that the change in $R(t)$ from time t to $t + \Delta t$ is proportional to $R(t)$. The constant of proportionality is $\{\textit{Inflow Rate} - \textit{Outflow Rate}\} \cdot \Delta t$.

If we subtract $R(t)$ from both sides and then divide by Δt, we get

$$\frac{R(t+\Delta t) - R(t)}{\Delta t} = \{\textit{Inflow Rate} - \textit{Outflow Rate}\} \cdot R(t)$$

Taking the limit of both sides as Δt goes to zero gives the derivative of $R(t)$:

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t} &= \{\textit{Inflow Rate} - \textit{Outflow Rate}\} \cdot R(t) \\ \frac{dR(t)}{dt} &= \{\textit{Inflow Rate} - \textit{Outflow Rate}\} \cdot R(t) \\ &= k \cdot R(t) \end{aligned}$$

where $k = \{\textit{Inflow Rate} - \textit{Outflow Rate}\}$. Hence, the rate equation for the reservoir $R(t)$ in the exponential system in Figure 2.9 is

$$\frac{dR(t)}{dt} = kR(t), \text{ where } k = \{\textit{Inflow Rate} - \textit{Outflow Rate}\} \tag{2.10}$$

It can be shown that the solution to this rate equation is

$$R(t) = R_0 e^{kt} \tag{2.11}$$

where R_0 = the value of $R(t)$ at time $t = 0$, and where k is given in Equation (2.10). This is left as an exercise for the reader.

The following facts about the rate constant k will aid in its interpretation. Figures 2.10a and 2.10b graphically illustrate these concepts.

Interpreting the Rate Constant k *in the Exponential System*

1. Recall from Equation (2.10) that $k = \textit{Inflow Rate} - \textit{Outflow Rate}$. Hence, k is the ***net growth (or decay) rate*** in the system.

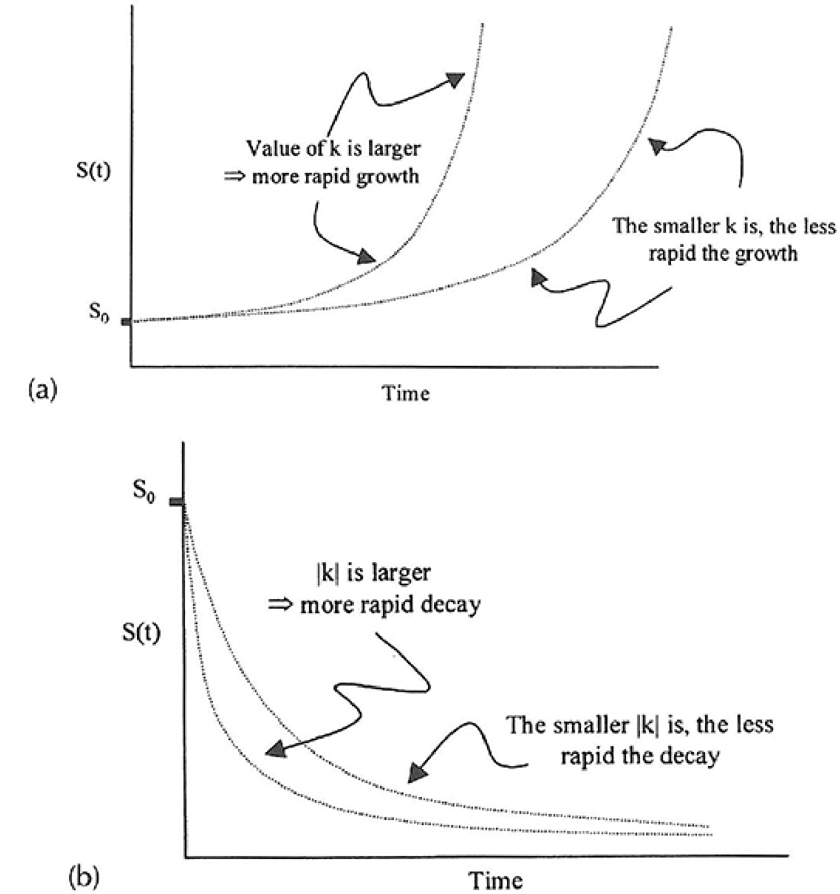


FIGURE 2.10. (a) Graphical interpretation of the rate constant k in an exponential growth system ($k > 0$). (b) Graphical interpretation of the rate constant k in an exponential decay system ($k < 0$).

2. If $k > 0$ then the system will exhibit ***exponential growth***.
3. If $k < 0$ then the system will exhibit ***exponential decay***.
4. The ***ratio of increase (if k is positive) or decrease (if k is negative)*** in the reservoir over one unit of time is e^k. In other words, after time advances one unit, the reservoir will change from $R(t)$ to $(e^k)\cdot R(t)$.
5. The ***larger*** $|k|$ is, the ***more rapid the growth or decay***.

Recall that if any reservoir $R(t)$ exhibits steady-state behavior after a time t_0, then the derivative of $R(t)$ will also be zero at all times after t_0. We know from Equation (2.10) that the rate equation for an exponential system is $\frac{dR(t)}{dt} = k \cdot R(t)$. It is easy to see that this derivative can be zero only if $k = 0$ or if $R(t) = 0$. Both of these situations are of no practical value because they correspond to the situation in which $R(t)$ is a constant value for all t (a very easy system to model!). With an exponential decay model, however, $R(t)$ does get closer and closer to zero the longer the system runs. As this happens, the

TABLE 2.4. Defining characteristics of exponential growth and decay.

Description	The reservoir increases at a rate that is proportional to its current size.
Rate equation	$\frac{dR(t)}{dt} = kR(t)$, where k is a constant. • k is the ***net growth rate*** or ***net decay rate*** of $R(t)$ • k = Inflow Rate − Outflow Rate • If $k > 0$ then the system will exhibit ***exponential growth*** • If $k < 0$ then the system will exhibit ***exponential decay*** • After time advances one unit, the reservoir will change from $R(t)$ to $e^k \cdot R(t)$. • The ***larger*** $\|k\|$ is, the ***more rapid the growth or decay***
Solution to the rate equation	$R(t) = R_0e^{kt}$, where R_0 is the initial value of $R(t)$ at time $t = 0$.
Graphical behavior	***Exponential growth***: Graph of $R(t)$ increases slowly at first and then more rapidly as time passes ***Exponential decay***: Graph of $R(t)$ decreases rapidly at first, and then more slowly as time passes; Graph of $R(t)$ eventually approaches a horizontal asymptote.
Steady state solution	Steady state of $\overline{R} = 0$ occurs only in the exponential decay system as $t \to \infty$.
Example applications	Population dynamics; heat transfer; fluid dynamics

derivative of $R(t)$ also gets very close to zero and the graph of $R(t)$ versus t approaches a horizontal asymptote. We will use the notation $\overline{R}$ to stand for the steady-state value that the reservoir $R(t)$ achieves (or approaches asymptotically). Hence, based on the preceding discussion, we can summarize these observations as follows:

An exponential system will never exhibit perfect steady-state behavior; however, if the system involves exponential decay, then $R(t)$ will asymptotically approach a steady-state value of $\overline{R} = 0$ the further out in time we go.

2.3.3 Exponential Growth or Decay: Summary

Table 2.4 summarizes important concepts for the exponential growth and decay system.

2.4 Behavior Pattern #3: Logistic Growth

2.4.1 Logistic Growth: Illustrative Example

An example graph of a logistic behavior pattern is given in Figure 2.11. It is clear from the graph why this behavior pattern is sometimes referred to as

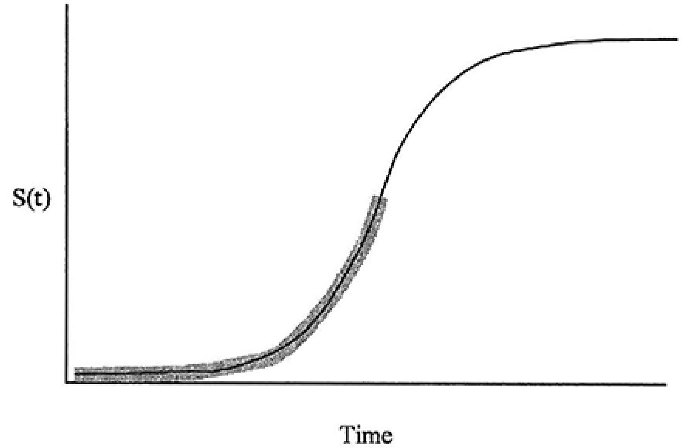


FIGURE 2.11. The logistic S-curve.

the "S-curve." An examination of Figure 2.11 suggests that a logistic system has some similarities with an exponential system. Notice in particular the section of the figure that is highlighted. This part of the curve looks like an exponential growth system. Unlike the exponential growth system, however, the growth in a logistic system eventually levels out and the system approaches a steady-state value. The underlying mechanisms that force this leveling off are what distinguish a logistic system from an exponential system.

Consider how a human population grows. Early on, whenever the resources necessary for survival are plentiful, and no other constraints on the reproductive choices of the people exist, the population grows exponentially. As time passes and the population gets very large, however, the resources available to support that population begin to be "stretched" to the point that further unchecked growth will lead to starvation, overcrowding, and disease. The death rate will likely begin to climb until it finally reaches a level that is the same as the birth rate. As this happens, the growth of the population will slow and eventually level off.

One way of describing the circumstances under which logistic behavior occurs is as follows. Logistic growth occurs whenever an exponential system is constrained so that the reservoir achieves a maximum level that is sustainable by the system. In such a case, the reservoir increases at a rate that is initially similar to an exponential system. As the reservoir approaches that maximum sustainable level, however, the rate of growth decreases and the system approaches a steady state.

2.4.2 Logistic Growth: System Features, Diagram, and Equations

A generic diagram for a logistic system is given in Figure 2.12. This Figure corresponds to the description given in Section 2.4.1. Notice from Figure

2.12 that the Inflow process operates in exactly the same way as in an exponential system: The size of the *Inflow* at any point in time is proportional to the current size of the reservoir. The *Outflow* process, however, does not operate the same way as in an exponential model. Notice that the size of the *Outflow* is determined by the equation

$$\textit{Outflow} = R(t) \cdot \textit{Unconstrained Growth Rate} \cdot \frac{R(t)}{\textit{Carrying Capacity}} \tag{2.12}$$

The *Carrying Capacity* converter in this model stands for the maximum size of the reservoir (i.e., the human population in our example) that can be sustained by the system. What is the proper value for the *Carrying Capacity*? This depends on the nature of the system that is being modeled. In our example involving a human population, the *Carrying Capacity* represents the maximum number of individuals that the system can support over the long term.

Equation (2.12) looks similar to the equation for the outflow in an exponential system. Recall that the outflow to an exponential system is calculated as:

$$\textit{Outflow} = R(t) \cdot \textit{Outflow Rate} \tag{2.13}$$

A comparison of Equation (2.13) with Equation (2.12) reveals that the logistic system calculates the Outflow by multiplying $R(t)$ by a "rate" that changes over time. That "rate" is given by the expression

$$\textit{Outflow Rate} = \textit{Unconstrained Growth Rate} \cdot \frac{R(t)}{\textit{Carrying Capacity}} \tag{2.14}$$

Consider how this rate will behave in our example involving a human population. When the population is small compared with the *Carrying Capacity*

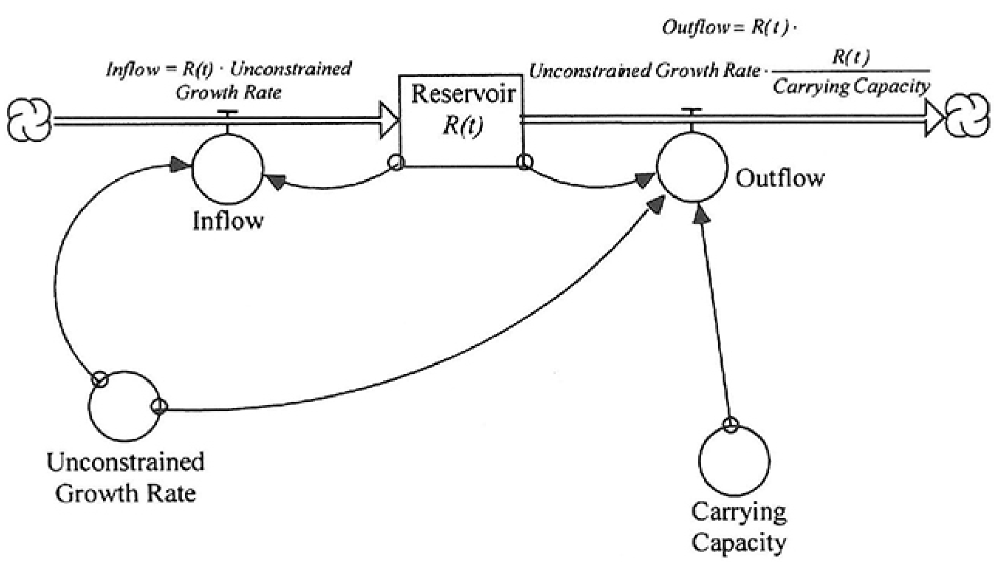


FIGURE 2.12. Generic system diagram for a logistic system.

of the system, the ratio $\frac{R(t)}{\textit{Carrying Capacity}}$ will initially be close to zero. Hence, the *Outflow Rate* given in Equation (2.14) will be very small. This means that the inflow (i.e., the "births") will exceed the outflow, and the system will grow exponentially; however, as the system progresses and the population approaches the *Carrying Capacity*, the ratio $\frac{R(t)}{\textit{Carrying Capacity}}$ will get closer to 1.0 and the *Outflow Rate* in Equation (2.14) will increase and approach the *Unconstrained Growth Rate*. Whenever this happens, the number of "deaths" will be very close to the number of "births," and the population's growth will slow down.

Notice also that the logistic system in Figure 2.12 includes both reinforcing and counteractive feedback loops. It will be left as an exercise to identify the feedback in this system.

The difference equation for the reservoir in Figure 2.12 is given by

$$\begin{aligned} R(t+\Delta t) &= R(t) + \{\text{Inflows} - \text{Outflows}\}\cdot\Delta t \\ &= R(t) + \{\textit{Unconstrained Growth Rate}\cdot R(t) - \textit{Outflow Rate}\cdot R(t)\}\cdot\Delta t \end{aligned} \quad (2.15)$$

The rate equation for the reservoir is given by

$$\frac{dR(t)}{dt} = k(t)\cdot R(t),$$

where

$$k(t) = \textit{Unconstrained Growth Rate}\cdot\left\{1 - \frac{R(t)}{\textit{Carrying Capacity}}\right\} \quad (2.16)$$

A comparison of Equation (2.16) with the rate equation for the exponential system Equation (2.10) reveals some important similarities between the exponential and logistic systems. Similar to the exponential system, the rate at which $R(t)$ changes is proportional to the current size of $R(t)$. Unlike the exponential system, however, the proportionality constant in the logistic system [i.e., $k(t)$] changes with time. In fact, if the initial value of $R(t)$ (i.e., R_0) is much smaller than the *Carrying Capacity*, then Equation (2.16) shows that $k(t)$ will be close to the *Unconstrained Growth Rate* and the system will initially behave like an exponential growth system. As time passes and $R(t)$ grows to values that approach the *Carrying Capacity*, $k(t)$ will approach zero, and the rate at which $R(t)$ grows will level off toward zero (no growth).

On the other hand, if R_0 is much larger than the *Carrying Capacity*, then $k(t)$ will start off with large negative values. Hence, $R(t)$ will begin to ***decrease*** rapidly toward the *Carrying Capacity*. As $R(t)$ shrinks to values near the *Carrying Capacity*, $k(t)$ will again approach zero, and $R(t)$ will shrink much more slowly until it finally levels off at the *Carrying Capacity*.

A common-sense understanding of the logistic system suggests that it will reach a steady state whenever the reservoir $R(t)$ approaches the *Carrying*

Capacity of the system. This is easily confirmed by examining the rate Equation (2.16). Recall that if the system reaches a steady state, then the derivative of $R(t)$ will be zero. That is, $R(t)$ reaches steady state if $\frac{dR(t)}{dt} = 0$, or, equivalently, if

$$\mathit{Unconstrained\ Growth\ Rate} \cdot \left(1 - \frac{R(t)}{\mathit{Carrying\ Capacity}}\right) \cdot R(t) = 0$$

This condition is achieved if and only if one of the following is true.

$$\mathit{Unconstrained\ Growth\ Rate} = 0, \text{ or}$$

$$R(t) = 0, \text{ or}$$

$$1 - \frac{R(t)}{\mathit{Carrying\ Capacity}} = 0$$

We can assume that the *Unconstrained Growth Rate* > 0 (i.e., the reservoir will grow if there is an unlimited *Carrying Capacity*). We can also assume that $R(t) > 0$ (the reservoir is not empty). Hence, the only possible way for $\frac{dR(t)}{dt}$ to be zero and remain zero for all later values of t, is if $1 - \frac{R(t)}{\mathit{Carrying\ Capacity}} = 0$ or, equivalently, $R(t) = \mathit{Carrying\ Capacity}$. Thus, we can see that the steady-state solution to the logistic system is $\overline{R} = \mathit{Carrying\ Capacity}$.

This makes sense whenever we consider that the *Carrying Capacity* represents the maximum reservoir size that can be sustained by the system. Based on the discussion in the previous paragraph, $R(t)$ will either grow or shrink toward this *Carrying Capacity*, depending on the size of the reservoir at the beginning.

The solution to the rate Equation (2.16) is

$$R(t) = \frac{\mathit{Carrying\ Capacity}}{1 + Ae^{-\text{Unconstrained Growth Rate}\cdot t}},$$

$$\text{where } A = \frac{\mathit{Carrying\ Capacity} - R_0}{R_0} \tag{2.17}$$

An analysis of Equation (2.17) confirms that $R(t)$ will behave so that it is driven to its *Carrying Capacity* value. Figure 2.13 illustrates this behavior for the case when R_0 is less than the *Carrying Capacity*. Make sure that you convince yourself that Figure 2.13 is consistent with the preceding description and that it matches the behavior you would expect from Equation (2.17).

2.4.3 Logistic Growth: Summary

Table 2.5 summarizes the logistic growth behavior pattern.

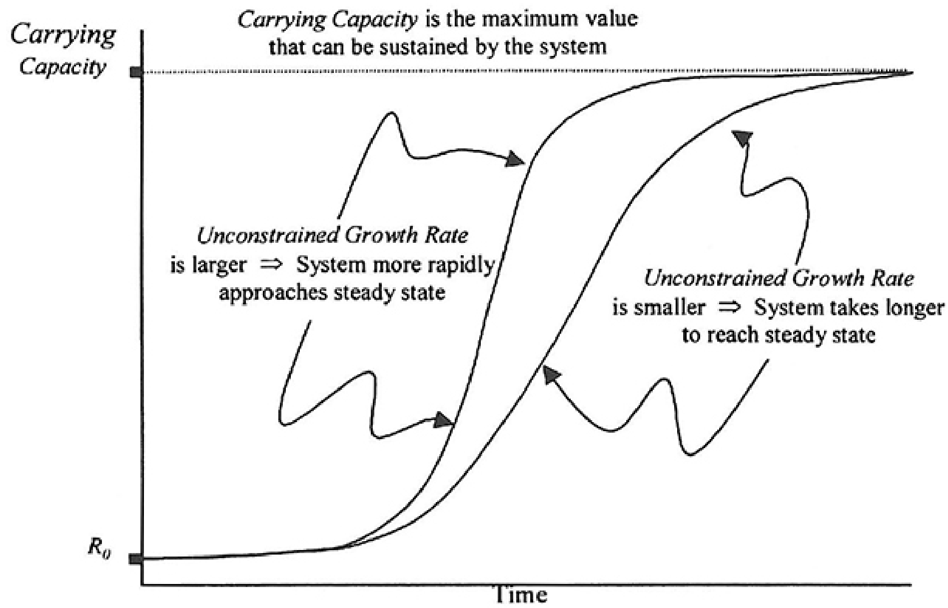


FIGURE 2.13. Graphical interpretation of the logistic system.

2.5 Behavior Pattern #4: Overshoot and Collapse

2.5.1 Overshoot and Collapse: Illustrative Example

Over the past several decades, there has developed an increased awareness of our culture's excessive consumption of petroleum. This nonrenewable resource (i.e., nonrenewable over any realistic time frame) is a source of energy and consumer products. The use of petroleum and petroleum products touches nearly every person's life in some significant way. If the global oil reserves should be completely used up, many life-critical products, services, and resources would be lost or jeopardized.

There are many historical examples of societies or animal species that have disappeared because of the loss of resources that were critical to their survival. Any system in which a population is dependent on a nonrenewable resource for survival is subject to a potential collapse as that resource is depleted. A population dependent on a *renewable* resource may also collapse if that resource is consumed at a rate much greater than the resource's rate of renewal. This latter situation is particularly true if the population's overconsumption of the resource reduces the overall ability of the resource to renew itself. An example might be cattle overgrazing in a field. If too many cattle graze on such a field, all the grass might be eaten, and the soil may be damaged to a point that the growth of new grass is limited. The new grass will only be able to support a smaller population of cattle than it would have initially.

We refer to this kind of system as an ***overshoot and collapse system***. Figure 2.14 illustrates the behavior of an overshoot and collapse system given a nonrenewable resource. Notice that as the population increases, the resource is consumed at a faster and faster rate. As the resource reaches dangerously low levels, the well being of the population is jeopardized. When this happens, the population begins to slow its growth, then eventually begins to decline until it collapses to a minimal level or disappears altogether.

TABLE 2.5. Defining characteristics of a logistic behavior pattern.

Description	Whenever the initial value R_0 is much smaller than what can be sustained over the long term, the system initially exhibits exponential growth, which flattens out as the reservoir approaches a maximum sustainable value (called the *Carrying Capacity* of the system). If the initial value is above the *Carrying Capacity*, then it will exhibit exponential decay, eventually approaching a stedy-state value equal to the *Carrying Capacity*.
Rate equation	$\frac{dR(t)}{dt} = k(t) \cdot R(t)$, where $k(t) = \textit{Unconstrained Growth Rate} \cdot \left\{1 - \frac{R(t)}{\textit{Carrying Capacity}}\right\}$ • The *Unconstrained Growth Rate* represents the rate at which $R(t)$ would *initially grow* (if resources were unlimited) • The larger the *Unconstrained Growth Rate*, the more rapidly the system will approach steady state. • *Carrying Capacity* represents the maximum value of $R(t)$ that the system can sustain over the long term.
Solution to the rate equation	$R(t) = \frac{\textit{Carrying Capacity}}{1 + Ae^{-\textit{Unconstrained Growth Rate} \cdot t}}$ where $A = \frac{\textit{Carrying Capacity} - R_0}{R_0}$
Graphical behavior	If $R_0 <$ *Carrying Capacity*, then $R(t)$ increases, eventually leveling off at the *Carrying Capacity*. If $R_0 <<$ *Carrying Capacity*, the early behavior of $R(t)$ will resemble exponential growth. If $R_0 >$ *Carrying Capacity*, then $R(t)$ decreases over time, eventually leveling off at the *Carrying Capacity*. If $R_0 >>$ *Carrying Capacity*, the early behavior of $R(t)$ will resemble exponential decay.
Steady state solution	$\bar{R}$ = *Carrying Capacity* (achieved only as $t \rightarrow \infty$)
Example applications	Population growth under limited resources; Epidemiology; Information and policy dissemination

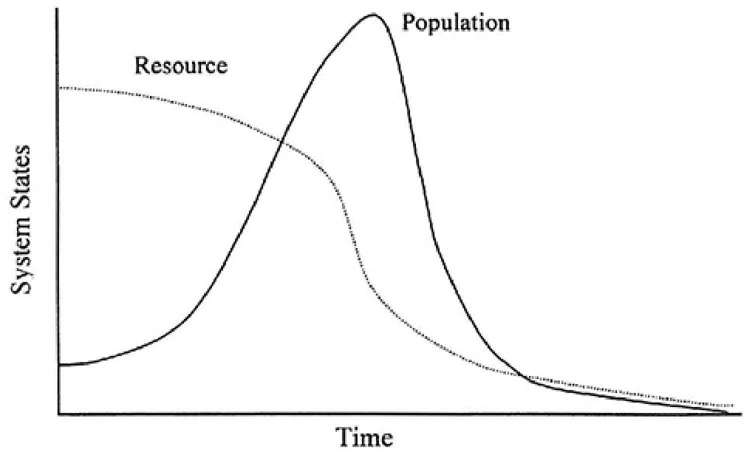


FIGURE 2.14. Example of overshoot and collapse behavior.

2.5.2 Overshoot and Collapse: System Features, Diagram, and Equations

Figure 2.15 displays an example diagram of an overshoot and collapse system. There are many different model constructs that will lead to similar behavior. The system diagram in Figure 2.15 was chosen for its overall simplicity and in order to illustrate the main features of this type of system. The use of the words *Population* and *Resource* in Figure 2.15 are not meant to imply that overshoot and collapse behavior can happen only with populations using resources. They are used here only for convenience. The

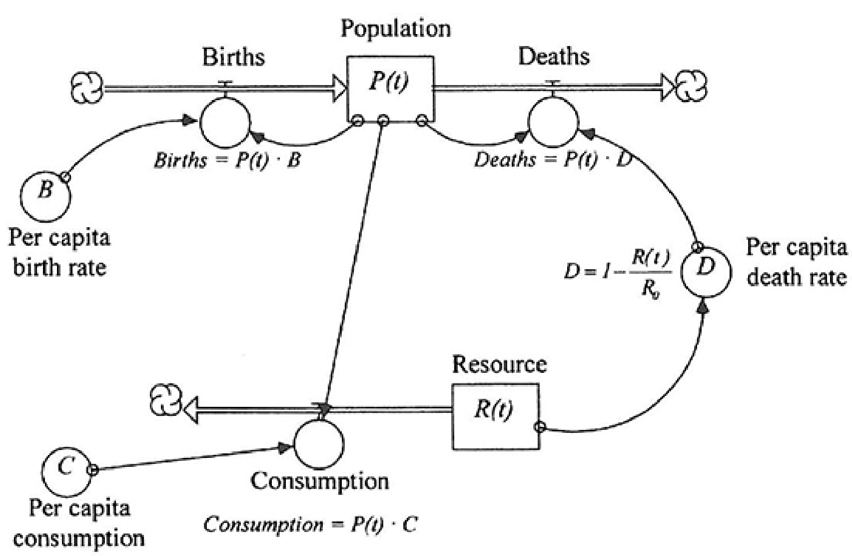


FIGURE 2.15. Generic system diagram for overshoot and collapse behavior.

exercises in this section provide an opportunity for you explore other overshoot and collapse systems.

Study Figure 2.15 very carefully to make sure you see how this system will lead to overshoot and collapse behavior. The following important characteristics are evident from the figure.

1. There are two reservoirs: the *Population* and the *Resource*.
2. Because the *Resource* has only an outflow process attached, it is nonrenewable; the system does not provide any means by which the *Resource* can be replenished.
3. Each individual in the *Population* consumes *C* *Resource* units in a single unit of time. Hence, as the *Population* size increases, the rate at which the *Resources* are consumed also increases. The value of the constant *C* depends on the particular system being modeled.
4. The size of the remaining *Resource* base affects the *Death Rate* in the *Population*. As the *Resource* base decreases, the *Death Rate* increases. The expression for relating the *Death Rate* to the *Resource* stock is defined in this example to be $\textit{Death Rate} = 1 - \frac{R(t)}{R_0}$ where $R(t)$ is the size of the *Resource* base at time t, and where R_0 is the initial size of the *Resource* base. This particular expression is somewhat arbitrary. Overshoot and collapse behavior will occur as long as the *Resource* is nonrenewable, and the *Death Rate* increases as the *Resource* decreases.
5. The system depicted in Figure 2.15 includes three different feedback loops. These are displayed in Figure 2.16 as well. Note that the loop involving the *Population* and *Births* and the loop involving *Population* and *Deaths* are reinforcing and counteractive feedback loops, respectively. The large loop involving *Population*, *Consumption*, *Resources*, and *Death Rate* is a counteracting feedback loop. This last loop is the feature in this system that leads to the ultimate collapse of the *Population*.

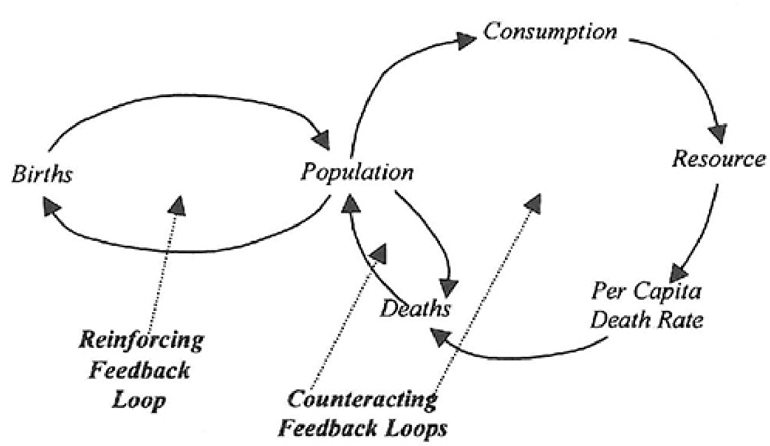


FIGURE 2.16. Feedback loops in the oveshoot and collapse model.

We can describe the circumstances under which the overshoot and collapse behavior pattern will occur as follows:

A system will exhibit overshoot and collapse behavior whenever one reservoir (i.e., a "population") depends on another nonrenewable reservoir (i.e. a "resource") for survival. As the population increases in size and overconsumes the resource, the resource becomes depleted to a point where the population cannot survive, leading to eventual collapse.

Using the same approach used for the linear, exponential, and logistic systems, the difference equations corresponding to Figure 2.15 can be derived. It is important to note that overshoot and collapse systems different from Figure 2.15 can be constructed, and those constructs would lead to a different set of difference equations. If you understand how this particular system and the associated equations work, however, you will be able to apply these same concepts to other overshoot and collapse constructs.

There are two reservoirs in Figure 2.15. Hence, we provide here two difference equations, one for each reservoir. By referring to Figure 2.15, the difference equation for the *Population*, is seen to be

$$P(t+\Delta t) = P(t) + \left\{B - \left(1 - \frac{R(t)}{R_0}\right)\right\} \cdot P(t) \cdot \Delta t \qquad (2.18)$$

Equation (2.18) shows that future values of the *Population* depend both on past values of $P(t)$ as well as on values of the *Resource* [$R(t)$]. We will elaborate on this relationship shortly. Equation (2.19) gives the difference equation for the *Resource* reservoir. Make sure that you see how this expression [and Equation (2.18)] are obtained from Figure 2.15.

$$R(t+\Delta t) = R(t) - P(t) \cdot C \cdot \Delta t \qquad (2.19)$$

This expression likewise indicates that future values of the *Resource* [$R(t)$] depend on past values of both the *Resource* and the *Population*.

Using Equations (2.18) and (2.19), we can develop the rate equation for each reservoir in this system. By rearranging terms in Equations (2.18) and (2.19) and taking the limit as Δt approaches zero, we get

$$\frac{dP(t)}{dt} = \left\{B - \left(1 - \frac{R(t)}{R_0}\right)\right\} \cdot P(t), \text{ and} \qquad (2.20)$$

$$\frac{dR(t)}{dt} = -C \cdot P(t) \qquad (2.21)$$

where B is the *per capita birth rate (per unit of time)*, and C is the *per capita consumption rate of the Resource base (per unit of time)*. The quantity R_0 is also fixed and stands for the value of $R(t)$ at time $t = 0$ (the initial size of the *Resource*).

Equations (2.20) and (2.21) are referred to as a ***coupled set of rate equations***. They are coupled because the rate at which the *Population* [$P(t)$]

changes [in Equation (2.20)] is a function of the *Resource* [$R(t)$]. Likewise, the rate at which the *Resource* changes [Equation (2.21)] is a function of the *Population*. That is, neither the *Population* nor the *Resource* can vary in size without affecting the rate at which the other changes.

A study of Equations (2.20) and (2.21) provides some important insights into the behavior of this system. From Equation (2.20) we see that the rate at which the *Population* changes is proportional to the current size of the *Population*; however, just as in the case with the logistic behavior pattern, the proportionality constant $\left[\text{i.e., the quantity } \left\{B-\left(1-\frac{R(t)}{R_0}\right)\right\}\right]$ varies over time. Early on, $R(t)$ will be close to its initial value R_0. Hence, $\left(1-\frac{R(t)}{R_0}\right)$ will be close to 0, and the proportionality constant $\left\{B-\left(1-\frac{R(t)}{R_0}\right)\right\}$ will be close to the per capita birth rate (B). Thus, early on, the *Population* will increase in a roughly exponential fashion with a rate constant close to the birth rate B. As time passes, more resources will be consumed and the value of $R(t)$ will eventually be much smaller than the initial value R_0. As this happens, the expression $\left(1-\frac{R(t)}{R_0}\right)$ will approach 1.0 and the proportionality constant will approach the value $B - 1$. Assuming that the per capita birth rate (B) is much less than 1.0 (a reasonable assumption), the quantity $\left\{B-\left(1-\frac{R(t)}{R_0}\right)\right\}$ in Equation (2.20) will be negative. This means that the *Population* will then decrease in size.

The negative sign on the right side of the rate equation for $R(t)$ [Equation (2.21)] indicates that $R(t)$ will always be decreasing at a rate that is proportional to the size of the *Population*. As the *Population* increases in size, $R(t)$ will decrease more rapidly. This matches our intuition: the larger the *Population*, the more rapidly the *Resource* base will be consumed.

Because the system involves two reservoirs, it is necessary that both reservoirs reach a steady state in order for the overall system to reach a steady state. Hence, the system reaches a steady state whenever the following conditions are met.

$$\frac{dP(t)}{dt} = 0 \text{ and } \frac{dR(t)}{dt} = 0 \qquad (2.22)$$

Setting the right-hand sides of Equations (2.20) and (2.21) equal to zero indicates that the system reaches steady state whenever

$$0 = \left\{B-\left(1-\frac{R(t)}{R_0}\right)\right\}\cdot P(t), \text{ and}$$
$$0 = -C\cdot P(t) \qquad (2.23)$$

It is clear that these conditions are satisfied under each of the following two cases:

$$\left\{B-\left(1-\frac{R(t)}{R_0}\right)\right\}=0 \text{ and } C=0, \text{ or}$$
$$P(t)=0$$

We will assume that the consumption rate C is greater than zero. In other words, we will assume that the *Population* does consume some of the *Resource* in each time unit. Hence, the first case is impossible. The second case corresponds to the situation in which the *Population* has totally collapsed and no longer exists. This happens only if the *Resource* is completely consumed so that the *per capita death rate* $\left(\text{i.e., } 1-\frac{R(t)}{R_0}\right)$ reaches 100%, thereby "killing off" the entire *Population*. This occurs asymptotically as we go further out in time. The system reaches steady state only in the limit as $t \to \infty$. The final steady-state values for the *Population* [$P(t)$] and the *Resource* [$R(t)$] are:

$$\overline{P}=0 \text{ and } \overline{R}=0 \tag{2.24}$$

By considering the rate Equations (2.20), (2.21), and (2.24) together, we can now predict how this system will behave. This description corresponds to Figure 2.14. At the beginning, the derivative for the *Population* [Equation (2.20)] will be positive; therefore, $P(t)$ will increase in size. As the *Population* increases, the *Resource* base $R(t)$ will decrease more and more rapidly. At some point, $R(t)$ will be so much less than its initial value R_0 that the derivative of the *Population* will become negative, and the *Population* will begin to drop. Both the *Population* and the *Resource* will asymptotically approach steady-state values of zero as we go further out in time.

2.5.3 Overshoot and Collapse: Summary

Table 2.6 summarizes the overshoot and collapse model characteristics.

2.6 Behavior Pattern #5: Oscillation

2.6.1 Oscillation: Illustrative Example

Many systems in the environment exhibit oscillatory behavior. For example, consider the cyclic behavior of the weather, the ocean tides, and the sun's energy output. The classic predator-prey relationship is another such example. Imagine a population of predators that live off of a renewable population of prey. As the number of prey increases, the number of predators also increases. As the number of predators increases, however, more prey

TABLE 2.6. Defining characteristics of an overshoot and collapse system.

Description	Two reservoirs. One is a nonrenewable "resource" and the other is a "population" that continually consumes the resource and also depends on it for survival.
Rate equations	For the Population: $\frac{dP(t)}{dt} = \left\{B - \left[1 - \frac{R(t)}{R_0}\right]\right\} * P(t)$ For the Resource: $\frac{dR(t)}{dt} = -C * P(t)$ • $P(t)$ = Population reservoir at time t • $R(t)$ = Resource reservoir at time t • B = per capita "birth" rate in the Population (per unit of time) • C = per capita consumption rate of the Resource (per unit of time) • R_0 = initial value of the Resource reservoir
Solution to the rate equation	Not provided
Graphical behavior	Initial exponential growth of the Population, followed by a peak and then collapse; Resource base continually decreases; Both approach a steady state after the system collapses.
Steady state solutions	$\bar{P} = \bar{R} = 0$ (achieved only as $t \to \infty$)
Example applications	Population growth with nonrenewable resources; epidemiology

are killed and consumed, thereby leading to a reduction in the prey population. As the prey population decreases, the predator population also decreases (because the predators cannot find as much to eat). This reduction in predators leads to a "rebound" in the number of prey (because there are not as many predators around to kill them). On and on this cycle goes, creating a sinusoidal pattern similar to the one in Figure 2.17.

2.6.2 Oscillation: System Features, Diagram, and Equations

The main feature of an oscillating system is the presence of a strong counteracting feedback loop that forces the system to oscillate around an equilibrium set of conditions. Consider the predator-prey example described earlier. Figure 2.18 illustrates this loop.

Figures 2.17 and 2.18 highlight several features that will cause a system to exhibit oscillatory behavior. These features will now be described.

1. The system contains at least two interdependent reservoirs. One reservoir can be thought of as the *Consumer* in the system, and the other can be thought of as the *Resource*. In some cases, the context will clearly identify

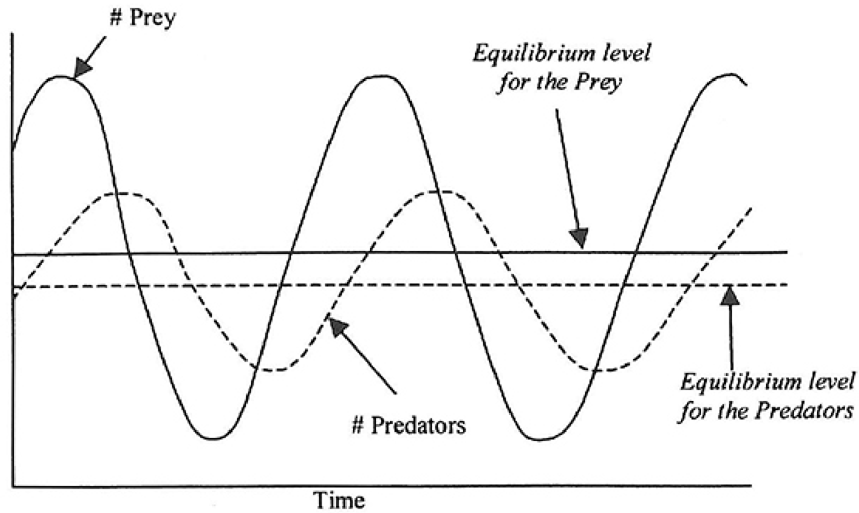


FIGURE 2.17. Typical predator–prey oscillatory behavior.

which role each reservoir plays. In many other cases, the designation of *Consumer* and *Resource* may be arbitrary.

2. The *Consumer* and *Resource* reservoirs have ***equilibrium values*** around which they oscillate.

3. The further one reservoir is from its equilibrium value, the more influence the other reservoir exerts to "pull it back" toward equilibrium. For example, whenever our example *Prey* reservoir is significantly above its equilibrium value, the *Predators* will grow rapidly and enthusiastically hunt down and kill the *Prey*, thereby driving them back toward equilibrium. If the *Predator* reservoir is significantly above its equilibrium level, then the *Prey* reservoir will rapidly shrink, thereby forcing the *Predators* back toward equilibrium.

Figure 2.19 gives a simplified example of an oscillatory system. Study this diagram to confirm that it includes the features described in the previous

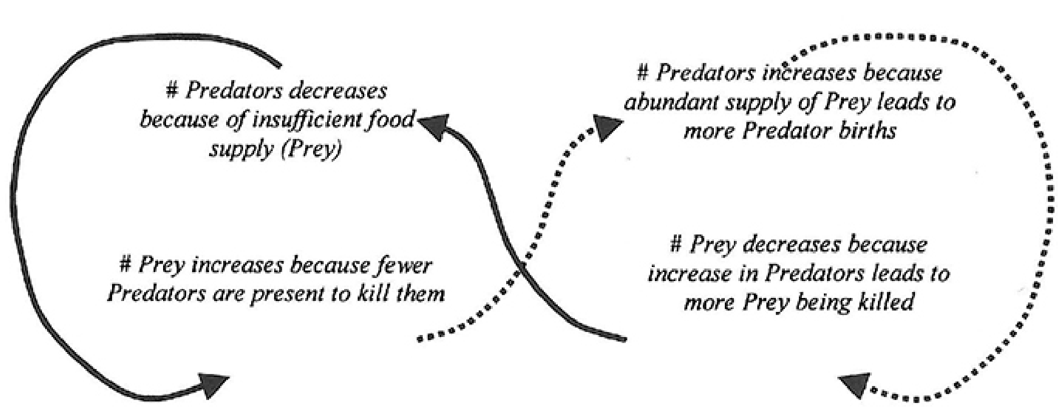


FIGURE 2.18. Counteracting feedback loop in the predator–prey example.

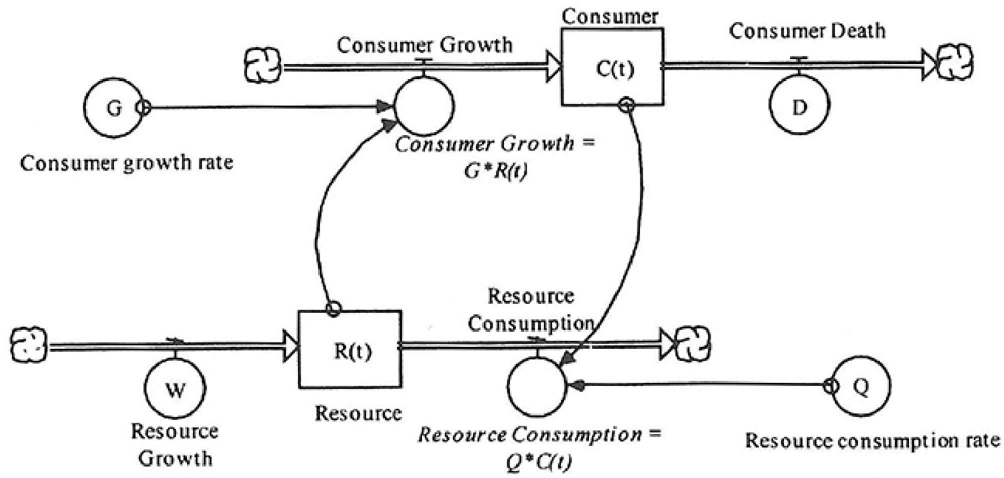


FIGURE 2.19. System diagram for a simple oscillating system.

paragraphs. In particular, note how the *Consumer* reservoir directly affects the rate at which the *Resource* reservoir is depleted (through the *Resource Consumption* outflow). Note also that the *Resource* reservoir affects the rate at which the *Consumer* reservoir grows (through the *Consumer Growth* inflow).

The difference equations for the two reservoirs in this system are given by

$$C(t+\Delta t) = C(t) + \{G \cdot R(t) - D\} \cdot \Delta t \tag{2.25}$$

$$R(t+\Delta t) = R(t) + \{W - Q \cdot C(t)\} \cdot \Delta t \tag{2.26}$$

We can rearrange terms in Equations (2.25) and (2.26) in the usual way and then take the limit as $\Delta t \to 0$ to get the following rate equations.

$$\frac{dC(t)}{dt} = G \cdot R(t) - D \tag{2.27}$$

$$\frac{dR(t)}{dt} = W - Q \cdot C(t) \tag{2.28}$$

A careful examination of Equations (2.27) and (2.28) will show that the two reservoirs will behave in some sort of cyclic pattern. For example, if the *Resource* reservoir is large, then Equation (2.27) indicates that the derivative of the *Consumer* reservoir will be a large positive value. Hence, the *Consumer* reservoir will rapidly grow in size. If the *Consumer* reservoir grows too large, however, then Equation (2.28) shows that the derivative of the *Resource* reservoir will turn negative, thereby indicating that the *Resource* will shrink in size. As $R(t)$ shrinks, the derivative Equation (2.27) will also get smaller and (eventually) turn negative, thereby indicating that the *Consumer* reservoir will no longer grow, but shrink instead. This will produce the kind of cyclic behavior described earlier. It can be shown that any system with the rate equations given in Equations (2.27) and (2.28) will oscillate around equilibrium values given by

$$\text{Equilibrium level for the Resource, } R(t), \text{ is } \frac{D}{G} \tag{2.29}$$

$$\text{Equilibrium level for the Consumer, } C(t), \text{ is } \frac{W}{Q} \tag{2.30}$$

The oscillatory system given here will reach a steady state whenever $\frac{dR(t)}{dt} = \frac{dC(t)}{dt} = 0$. This occurs whenever both reservoirs are equal to their equilibrium values given in Equations (2.29) and (2.30). Hence, the steady-state solution to the simple oscillatory system in Figure 2.19 is given by

$$\overline{R} = \frac{D}{G} \text{ and } \overline{C} = \frac{W}{Q} \tag{2.31}$$

Notice that the system reaches steady state only if the reservoirs are equal to their equilibrium values at the same time. For example, if only *R(t)* is at the equilibrium value, then it will not be "allowed" to stay at equilibrium. If the *Consumers* are above their equilibrium, then they will consume too many of the *Resources* and drive the *R(t)* reservoir below equilibrium. If the *Consumers* are below their equilibrium value, then the *Resources* will begin to proliferate (due to their being too few *Consumers* to keep the population in check).

2.6.3 Oscillation: Summary

Table 2.7 summarizes the important characteristics of oscillating systems.

2.7 Exercises

Section 2.2

1. Consider a new landfill in which solid waste is deposited at a rate of 25 metric tons/day. Draw a system diagram depicting this system. Fill in constant values for each of the quantities in the system. Make a sketch showing the volume of waste in the landfill as a function of time. Write down the rate equation for the volume of waste in the landfill and specify the value of the rate constant.

2. Define three systems in real life that will exhibit linear growth or decay. For each example, identify the following: (a) The reservoir that will exhibit this linear dynamic (specify its units); (b) All of the inflows and outflows

TABLE 2.7. Defining characteristics of a simple oscillating system.

Description	Two interdependent reservoirs (i.e., a Consumer reservoir and a Resource reservoir). The Consumer "consumes" Resource units. The Resource enables the Consumer to grow. Both oscillate around equilibrium values. The further one reservoir is from its equilibrium value, the more the system "works" to drive it back toward equilibrium.
Rate equations (refer to Figure 2.17)	For the Consumer $\frac{dC(t)}{dt} = G{\cdot}R(t) - D$ For the Resource: $\frac{dC(t)}{dt} = W - Q^*(t)$ • G = Consumer Growth Rate (reflect how the growth of the Consumer reservoir depends on the size of the Resource) • D = Consumer Deaths per time unit • W = Resource Growth per time unit • Q = Resource Consumption Rate (reflects how the depletion of the Resource depends on the size of the Consumer reservoir)
Solution to the rate equation	Not provided
Graphical behavior	Graph of each reservoir resembles a sinusoidal wave. Each wave oscillates about an equilibrium value. The equilibrium value for the Consumer is $\frac{W}{Q}$. The equilibrium value for the Resource is $\frac{D}{G}$. The period and amplitude of each sine wave is determined by D, G, W, Q, and the initial values of the reservoirs R_0, and C_0.
Steady state solutions	The system will run at steady state only if both reservoirs begin at their equilibrium values. That is, the steady state is reached at $\overline{C} = \frac{W}{Q}$ and $\overline{R} = \frac{D}{G}$.
Example applications	Predatory–Prey systems; Systems with consumers and renewable resources

(specify units); (c) Whether the system will typically exhibit linear ***growth*** or linear ***decay***.

3. Assume that the reservoir in Figure 2.3 represents the mass of a particular pollutant in a lake at time *t*, where *t* is measured in years. Assume the initial pollutant mass in the lake is 1,000 kg. Specify the units for each of the flow processes in this model.

4. You will find nine different sets of values for the flows in the Figure 2.3 listed in Table 2.8. For each case (a) Specify whether the resulting system will exhibit linear growth, linear decay, or neither (assume that time is measured in minutes); (b) If the system exhibits linear growth or decay, determine the value of the rate constant k and write down the equation [corresponding to equation (2.7)] for the reservoir value $R(t)$ that is the solu-

TABLE 2.8. List of data for exercise 4.

Case	Inflow 1	Inflow 2	Inflow 3	Outflow 1	Outflow 2
A	10	20	35	35	20
B	10	20	35	35	35
C	10	20	sin(t)	30	sin(t)
D	10	20	sin(t)	30	0
E	5t	0	0	5t	0
F	10	0	0	0	0
G	0	0	0	0	10
H	10	20	sin(t)	0.75sin(t)	0.25sin(t)
I	10	20	sin(t)	0.75sin(t)	0

tion to the rate equation in Table 2.3. Make sure you give numeric values for all quantities.

Section 2.3

5. Use STELLA® to construct the mouse population example system in Figure 2.6. Assume that the system starts with two mice. Try different values for the *Birth Rate* and *Death Rate* and run the model to see what effect they have on the system. Under what conditions does the mouse population increase over time? When does it decrease over time? When is the population at a steady state?

6. Define three systems in real life that will exhibit exponential growth or decay. For each example, identify the following: (a) The reservoir that will exhibit this type of behavior (specify its units); (b) All of the inflows and outflows (specify units); (c) Whether the system will typically exhibit exponential ***growth*** or exponential ***decay***.

7. Briefly describe what factor(s) you think would determine the value of the "flow rate" converter in Figure 2.8.

8. Explain why the feedback loop involving the reservoir and outflow process in Figure 2.9 is a counteracting feedback loop.

9. Show mathematically that the solution to the exponential rate Equation (2.10) is the expression given in Equation (2.11). [Hint: Take the derivative of the expression in Equation (2.11) and show that it is equal to the right-hand side of Equation (2.10)].

10. In STELLA®, modify the model in Figure 2.8 to include a constant water inflow of 10 cc/sec and do the following: (a) Write down the difference equation for the water volume in the bucket; (b) Derive the rate equation for the water volume in the bucket in this new system; (c) Will this new system behave in an exponential fashion? Under what conditions?

11. Is it reasonable to assume that populations of people or other living organisms will exhibit exponential behavior indefinitely? Why or why not?

12. Consider what happens whenever a 90°F can of soft drink is placed in a 38°F refrigerator. The can's temperature will immediately begin to drop

and will rapidly approach the ambient temperature in the refrigerator. As the can's temperature gets closer to 38°F, however, the rate at which its temperature drops will decrease until (after a long time) the temperature of the can is virtually identical to the temperature inside the refrigerator. Do the following: (a) Sketch a graph of the can's temperature versus time; (b) Draw a system diagram for modeling the temperature of the can. Use a reservoir to represent the can's temperature. Note that the rate at which the can's temperature changes is proportional to the difference between the can's temperature and the ambient temperature in the refrigerator.

Section 2.4

13. Assume that the Reservoir in Figure 2.12 represents a population of people and that time is measured in years. Specify the units for all the system elements in Figure 2.12.

14. Define three systems in real-life (other than those described in this chapter) that will exhibit logistic growth. For each example, identify the following:

- The reservoir that will exhibit this type of behavior (specify its units)
- All of the inflows and outflows (specify units)

15. Use STELLA®, to construct the logistic model Figure 2.12. Assume that the reservoir represents a population of people and that time is measured in years. Run the model for 100 years with each of the following sets of values for R_0, the *Unconstrained Growth Rate*, and the *Carrying Capacity*. In each case, make a graph of the reservoir versus time and explain why the system behaves as it does. In addition, for each case, write down Equation (2.17), giving numerical values for all the parameters in that expression.

- $R_0 = 10$, Unconstrained Growth Rate = 0.1, Carrying Capacity = 1,000
- $R_0 = 10$, Unconstrained Growth Rate = 0.5, Carrying Capacity = 500
- $R_0 = 2{,}000$, Unconstrained Growth Rate = 0.1, Carrying Capacity = 500
- $R_0 = 2{,}000$, Unconstrained Growth Rate = 0.5, Carrying Capacity = 1,000

16. Sketch a graph like Figure 2.13 showing what the system will behave like if the initial value of the reservoir is greater than the *Carrying Capacity* of the system. Make sure you show how the value of the *Unconstrained Growth Rate* will affect the shape of the graph.

17. Identify and describe the feedback loops in Figure 2.12. Specify whether each is positive or negative feedback.

18. Show that Equation (2.16) is the rate equation for the logistic system in Figure 2.12. [Hint: Begin with the difference equation in Equation (2.15). Then use an approach similar to what was used to derive the rate equation for the exponential system, Equation (2.10)].

19. In the logistics model described in Section 2.4, we assumed that the "death" rate would increase and the birth rate would remain constant as the population approached the Carrying Capacity of the system. In more advanced cultures, however, people may adopt a more proactive approach before resources are seriously depleted. In this scenario, individuals in the population may choose to have fewer children, thereby reducing the overall birth rate. In this case, the birth rate will decrease over time and the death rate will remain constant. The growth should again level off as the population approaches the Carrying Capacity. Create a systems model in STELLA®, to match this scenario. Make sure you specify all the equations and units for calculating the system quantities specify number using for the initial size of the population, the carrying capacity and any other values necessary to run the model. Include a graph that shows the Population and Resource reservoirs over time. Derive the rate equation for the system. This equation should be similar to (but not exactly the same) as Equation (2.15). Run the model for 50 years.

Section 2.5

20. Assume that the *Resource* reservoir in Figure 2.15 is measured in generic "resource units." In addition, assume that time is measured in years. What are the units for the *Consumption* flow process?

21. Assume that in Figure 2.15 the initial value of Resource is 10,000 units, the initial value of Population is 10 people, the birth rate is 0.50 births per capita per year, and each person consumes two resource units per year. Use these values and Figure 2.15 to construct the overshoot and collapse model in STELLA®. Make a graph showing the *Population* and *Resource* reservoirs over time. Run the model for 50 years.

a. Decrease the *Birth Rate* to 0.1 and rerun the model. Make sure you run long enough for the system to stablize. What differences do you see from the case in which the *Birth Rate* is 0.5? Make sure you pay attention to the scale on the vertical axis of the graph! Can you explain why the system's behavior changes in this way?
b. Change the *Birth Rate* back to 0.50 and change the *Consumption Rate* to 20 units/person/year. Rerun the model and compare the results with the original case. Can you explain why the system's behavior changes in the ways that it does?
c. Summarize what you have learned from these experiments with the model. What conditions lead to more rapid collapse of the system?
d. See if you can identify values for the *Birth Rate* and *Consumption Rate* that will not lead to a total collapse of the *Population* for the preceding exercise. Can this be done? If so, how? Why is this the case?

22. The expression used for the *Per Capita Death Rate* in Figure 2.15 is $D = 1 - \frac{R(t)}{R_0}$. Under this formulation, the *Death Rate* will eventually reach

levels that are very close to 100%. It may be the case, however, that the *Population* is not completely dependent on the *Resource* for survival. Whenever this is so, then the *Death Rate* may not approach 100% as the *Resource* is depleted. Suppose that the *Death Rate* D will reach only 60% as $R(t)$ approaches zero. How could you modify the expression for D to accommodate this fact? Incorporate this modification into the model corresponding to Figure 2.15 and then run the model with the original values for the *Birth Rate, Population,* and *Resource,* given in Exercise 21. How does the system behave? Can you explain why?

23. Define three systems in real life (other than the oil consumption example described in this section) that can exhibit overshoot and collapse behavior. Make sure that you sketch a system diagram for each case. Clearly identify all reservoirs, flows, and converters by giving them descriptive names. Clearly define the units for each quantity in the system.

24. Some systems involve a *Resource* that can be replenished. For example, the *Resource* might represent a food supply that is renewable through agricultural methods or natural growth. Modify your STELLA® model of the original system described in Exercise 21 so that the *Resource* can be renewed via a *Renewal* flow process. Assume that the *Renewal* inflow is proportional to the size of the *Resource* (as in an exponential model). Also assume that a converter called the *Renewal Rate* represents the proportionality constant. Set the *Renewal Rate constant* to be equal to 0.05. How does the behavior of this new system compare with the behavior of the original system? Can you explain why?

Section 2.6

25. Assume that time is measured in hours in Figure 2.19, that the *Consumer* reservoir is expressed as a number of organisms, and that the *Resource* reservoir is expressed as the number of generic resource units. Specify the units for all the other quantities in the model.

26. Use STELLA® to build the model in Figure 2.19. Populate the model with the values specified later. Make a graph showing both the *Consumer* and the *Resource* reservoirs over time. Run the model for 25 years, and answer the following questions. The graph of the *Consumer* and *Resource* reservoirs should exhibit simple oscillatory behavior. [Note: The oscillatory model has a complicated enough behavior that we must take special care to avoid serious round-off errors in STELLA®. This is done by using a very small time step and by using a more sophisticated numerical algorithm. You can make these changes by selecting ***Time Specs*** under the ***Run*** menu. The value of ***DT*** specifies the time step. Change ***DT*** to ***0.0625*** hours. In addition, instruct STELLA® to use the ***Runge-Kutta 2*** method of integration in the ***Time Specs*** window of STELLA®.]

$$G = 1 \quad D = 20 \quad Q = 1 \quad W = 15 \quad C_0 = 10 \quad R_0 = 15$$

a. What are the ***equilibrium values*** for the *Consumer* and *Resource* reservoirs? Note that the equilibrium values are those midpoint values around which each reservoir oscillates.
b. What is the ***period*** of this system? (A *period* is the length of time it takes for the system to complete one full cycle).
c. When is the first time that the *Consumer* reservoir reaches its equilibrium value in the simulation run? Where is the *Resource* reservoir in relationship to its equilibrium value during this same time? How does the level of the *Resource* affect the behavior of the *Consumer* from this point in time until the *Consumer* next reaches its equilibrium?
d. Do the *Consumer* and the *Resource* ever achieve equilibrium at the same time? What do you think would happen if they did?
e. Change the value of *W* to 20 and re-run the model. What has changed? What explanation can you give?
f. Change *W* back to 15 and then change the value of *G* to 0.5 and re-run the model. What has changed? What explanation can you give?
g. Now change *G* back to 1. Set the starting value of $C(t)$ and $R(t)$ to be $C_0 = 15$ and $R_0 = 20$. Run the model. Can you explain why the system behaves this way?

27. Imagine that you are a wildlife manager, responsible for keeping the population of a certain deer species as close to equilibrium as possible. Suppose that the only predators in your system are a population of wolves. How might knowledge of the steady-state behavior of this system aid you in your job? What would you do to try and maintain the deer population to a level as close to steady state as possible?

Model Building Exercise

28. Construct a system model in STELLA® that describes the relationships between the oil supply and the population of fossil fuel–burning vehicles. Your model should include at least three reservoirs: (1) the fossil fuel vehicles, (2) the supply of processed oil that is ready for consumption, and (3) the underground reserves of crude oil. Your model should include at least one feedback loop. Include flow processes for the production and obsolescence of vehicles and for the production and consumption of processed oil. Specify all interrelationships and equations or graphical relationships that are necessary in order for the model to run (apart from the values of constants). Provide descriptions within STELLA® of each of the system components. Specify their units. Prepare a brief (one- or two-page) write-up describing your system and giving the rationale for the system diagram. Include a copy of the system diagram and a graph of the model output as part of the write-up. Answer the following questions:

a. Describe the feedback loops in your system. Are they examples of positive or negative feedback?
b. Which types of behavior patterns will your model exhibit (linear, exponential, logistic, overshoot and collapse, or a combination)? Under what conditions will the type of behavior you identified occur? Briefly justify your answer.

2.8 Suggested Readings

Haberman, R. 1977. *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow*. New Jersey: Prentice-Hall, Inc.

High Performance Systems. 1996. *An Introduction to Systems Thinking*. Hanover, NH: High Performance Work Systems, Inc.